LINEAR OPERATORS IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

BY

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1. **Introduction.** The taking of the real part of an analytic function of one complex variable is an operation which transforms (in function space) the totality of these functions into the totality of harmonic functions of two variables. Almost every theorem on analytic functions gives rise to a corresponding theorem in the theory of the latter functions. The similarity in structure suggests the use of an analogous approach in the theory of functions satisfying linear partial differential equations of the elliptic type,

$$(1.1)\quad L(U) \equiv U_{z\bar z} + a(z, \bar z)U_z + b(z, \bar z)U_{\bar z} + c(z, \bar z)U = 0,$$

$$U_{z\bar z} = \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)\Big/4, \qquad U_z = \left(\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}\right)\Big/2,$$

$$U_{\bar z} = \left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right)\Big/2, \qquad z = x + iy, \qquad \bar z = x - iy.$$

In this connection there arises first of all, the question of finding all operators of this kind. *All operators* which transform the class of functions $f(z)$ into the class of functions $U(z, \bar z)$, $L(U)=0$, (functions of both classes considered in a sufficiently small neighborhood of the origin) can be determined by formal calculations.

However the transformation of various results requires that the operation be applicable "in the large," that is to say, that every analytic function f regular in a domain $\mathfrak{B}^2$ of a certain class $\mathfrak{D}$ may be transformed into a function U regular in $\mathfrak{B}^2$ and that the inverse operator $U = \mathrm{P}^{-1}(f)$ possess the same property.

Further, for many purposes it is important that for various sequences of functions f_n, the relation $\lim_{n\to\infty} \mathrm{P}[f_n(z)] = \mathrm{P}[\lim_{n\to\infty} f_n(z)]$ shall hold([1]).

In addition to the problem of studying all operators and their classification from this point of view, one may consider a particular operator. To a

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([1]) A system of functions $\{\varphi_\nu(z)\}$, $\nu = 1, 2, \cdots$ possessing the property that every function u regular in a domain $\mathfrak{B}^2$ of certain type $\mathfrak{D}$ can be approximated therein by $\sum_{\nu=1}^{n} \alpha_{\nu}\varphi_\nu(z)$, may be denoted as a basis of the class of analytic function with respect to $\mathfrak{D}$.

An operator with the above properties transforms a basis $\{\varphi_\nu(z)\}$ into a basis $\{\mathrm{P}[\varphi_\nu(z)]\}$ of the class of functions U with respect to $\mathfrak{D}$. Certain properties of the basis $\{\mathrm{P}(z^\nu)\}$ can be used to characterize the operation P.

certain extent it is useful first to study the latter problem, in order to see how the different properties of the operator influence the transformation of the results, and in order to get a clearer concept of the laws which govern this transformation.

In this paper we shall study the second question. We shall return to the first problem at another place.

NOTATION. We denote the cartesian coordinates of the plane by x, y. Often, however, we shall write $z=x+iy$, $\bar{z}=x-iy$, instead of x and y. We note that if we extend the functions considered to complex values of x and y, the variables z and $\bar{z}$ are no longer conjugate to each other.

Manifolds will be denoted by German letters, the upper index indicating the dimension of the manifold.

$\mathfrak{F}^2$ will always denote a star-domain of the (x, y)-plane, with center at the origin. Its boundary will be denoted by $\mathfrak{f}^1$. $\mathfrak{f}^1$ is supposed to be a differentiable curve.

Further, we denote by $\mathrm{E}[\cdots]$ the set of points whose coordinates satisfy the relations indicated in brackets. $\mathfrak{S}$ means the logical sum.

I. THE OPERATOR AND ITS PROPERTIES

2. **The class of functions $\mathcal{C}(\mathrm{E})$. A survey of obtained results.** A complex harmonic function $h(z, \bar{z})$ of two real variables x, y can be represented in the form

$$h(z, \bar{z}) = F(z) + G(\bar{z}) \tag{2.1}$$

where $F(z)$ and $G(z)$ are analytic functions of one complex variable. Since we can write $F(z)=\int_{-1}^{1} f[(z/2)(1-t^2)](1-t^2)^{-1/2}dt$, (2.1) can be written in the form

$$h(z, \bar{z}) = \int_{-1}^{1} \{f[(z/2)(1-t^2)] + g[(\bar{z}/2)(1-t^2)]\}(1-t^2)^{-1/2}dt \tag{2.2}$$

where f and g are analytic functions, of one complex variable, which are regular at the origin.

As was indicated in [3] (numbers in brackets refer to the bibliography), the representation (2.2) can be generalized. Suppose that a, b, c, are analytic functions of two complex variables z, $\bar{z}$. Then for every equation $L(U)=0$ [see (1.1)], there exist functions

$$\mathrm{E}_k(z, \bar{z}, t) = 1 + t^2 z\bar{z}\mathrm{E}_k^*(z, \bar{z}, t), \qquad k = 1, 2, \tag{2.3}$$

such that every solution of $L(U)=0$ can be written in the form

$$\begin{aligned} U(z, \bar{z}) = \int_{-1}^{1}\Big[&\exp\Big(-\int_0^{\bar{z}} a\,d\bar{z}\Big)\mathrm{E}_1(z, \bar{z}, t)f((z/2)(1-t^2)) \\ &+ \exp\Big(-\int_0^{z} b\,dz\Big)\mathrm{E}_2(z, \bar{z}, t)g((\bar{z}/2)(1-t^2))\Big](1-t^2)^{-1/2}dt. \end{aligned} \tag{2.4}$$

For many purposes, instead of considering the functions

$$U(z, \bar{z}) = \exp\left(-\int_0^{\bar{z}} a d\bar{z}\right)\int_{-1}^{1} \mathrm{E}_1(z, \bar{z}, t)f((z/2)(1 - t^2))(1 - t^2)^{-1/2}dt,$$

it is useful to investigate the functions

$$(2.5)\quad \begin{aligned} u(z, \bar{z}) &= U(z, \bar{z})\cdot\left[\exp\left(\int_0^{\bar{z}} a d\bar{z}\right)\right] \\ &= \int \mathrm{E}_1(z, \bar{z}, t)f((z/2)(1 - t^2))(1 - t^2)^{-1/2}dt. \end{aligned}$$

Let $\mathcal{C}(1)$ be the totality of analytic functions of the complex variable z which are regular at the origin. The totality of functions $u(z, \bar{z})$ which can be represented in the neighborhood of the origin in the form

$$(2.6)\quad u(z, \bar{z}) = \mathrm{P}(f) = \int_{-1}^{1} \mathrm{E}(z, \bar{z}, t)f((z/2)(1 - t^2))(1 - t^2)^{-1/2}dt, \qquad f \in \mathcal{C}(1),$$

will be known as the class([2]) $\mathcal{C}(\mathrm{E})$. We define $\mathrm{E}(z\ \ \bar{z}, t)$ to be *the generating function of* $\mathcal{C}(\mathrm{E})$, f *the associate of* u, and call the domain in which the representation is valid *the domain of association.*

If E_1 satisfies a certain partial differential equation, the functions, $U \in [\exp(-\int_0^{\bar{z}} a d\bar{z})]\cdot\mathcal{C}(\mathrm{E}_1)$ satisfy the equation $L(U) = 0$. [$f(z, \bar{z})\cdot\mathcal{C}(\mathrm{E}_1)$ denotes here the class of functions $f(z, \bar{z})\cdot u$, where $u \in \mathcal{C}(\mathrm{E}_1)$.] The totality of the solutions of $L(U) = 0$ is given by

$$\left[\exp\left(-\int_0^{\bar{z}} a d\bar{z}\right)\right]\cdot\mathcal{C}(\mathrm{E}_1) + \left[\exp\left(-\int_0^{z} b dz\right)\right]\cdot\mathcal{C}^*(\mathrm{E}_2)$$

where $\mathcal{C}^*(\mathrm{E})$ is a class analogous to $\mathcal{C}$, the associates of whose functions are analytic functions of $\bar{z}$. The present paper is devoted to a general study of the functions of any class $\mathcal{C}(\mathrm{E})$, that is, a class of analytic functions of two real variables x, y which can be represented in a sufficiently small neighborhood of the origin by the right-hand member of (2.6)([3]).

In this paper we drop the assumption that the functions $u \in \mathcal{C}(\mathrm{E})$ satisfy

([2]) We may also consider classes of functions for which a representation analogous to (2.6) holds in the neighborhood of a point a, $a \neq 0$. Functions satisfying $L(U) = 0$ possess the property that the representation (2.4) exists for every point a. The study of the dependence upon the point a of $\mathrm{E}(z, \bar{z}, t|a)$ and the associate $f(z|a)$, of a function u is an interesting problem of the theory.

([3]) The relation (2.6) may be interpreted as a mapping (in the function space) of $\mathcal{C}(1)$ into the class $\mathcal{C}(\mathrm{E})$. We are going to study the duality between the theories of the functions of $\mathcal{C}(1)$, and those of $\mathcal{C}(\mathrm{E})$.

Note that our space of functions includes those which are not defined *in one fixed domain, but only in a sufficiently small neighborhood of the origin.*

These functions arise also in other connections, for example, as a set of particular solutions a linear partial differential equation. We suppose only that the functions u possess the two properties A and B which we now describe.

A. The function E can be written in the form

$$\mathrm{E}(z, \bar{z}, t,) = 1 + t^2 z\bar{z}\mathrm{E}^*(z, \bar{z}, t), \tag{2.7}$$

where E^* is an analytic function of two *complex* variables z, $\bar{z}$ regular in the region $\mathrm{E}[|z| < \infty, |\bar{z}| < \infty]$ and a continuously differentiable function of z, $\bar{z}$, t in $\mathrm{E}[|z| < \infty, |\bar{z}| < \infty, |t| \leq 1]$.

We note that from A follow:

A_1. Every $u \in \mathcal{C}(\mathrm{E})$, regular in a star-domain $\mathfrak{F}^2$, can be continued analytically in[4]

$$\mathfrak{R}^4(\mathfrak{F}^2) = \mathrm{E}[z = a + ib, \bar{z} = a - ib, (a, b) \in \mathfrak{F}^2, a, b \text{ real}]. \tag{2.8}$$

A_2. For every $u \in \mathcal{C}(\mathrm{E})$,

$$| u(z, \bar{z}) | \leq c \max_{\zeta \in \mathfrak{F}^2} | f(\zeta) |, \qquad \text{for } (z, \bar{z}) \in \mathfrak{R}^4(\mathfrak{F}^2) \tag{2.9}$$

and

$$c = \max_{(z, \bar{z}) \in \mathfrak{R}^4(\mathfrak{F}^2), |t| \leq 1} | \mathrm{E}(z, \bar{z}, t) |.$$

B. There exists, for every $\mathfrak{F}^2$, an operator $G(z, \bar{z}, \zeta, \bar{\zeta}, X_{00}, \cdots, X_{mn})$ such that

$$u(z, \bar{z}) = \int_{\mathfrak{f}^1} G[z, \bar{z}, \zeta, \bar{\zeta}, u(\zeta, \bar{\zeta}), u_\zeta(\zeta, \bar{\zeta}), \cdots, u_{\zeta^m \bar{\zeta}^n}(\zeta, \bar{\zeta})] ds_\zeta, \quad (z, \bar{z}) \in \mathfrak{R}^4(\mathfrak{F}^2). \tag{2.10}$$

Here $\mathfrak{f}^1$ is the boundary of $\mathfrak{F}^2$, ds_ζ is the line element of $\mathfrak{f}^1$, and $G(z, \bar{z}, \zeta, \bar{\zeta}, X_{00} \cdots X_{mn})$, $[|X_{pq}| < \infty, (pq) = (00), \cdots, (mn)]$ is an analytic function of two complex variables z, $\bar{z}$, which is regular in $\mathfrak{R}^4(\mathfrak{F}^2)$.

Since we suppose that $\mathrm{E}(z, \bar{z}, t)$ is an analytic function of two complex variables z, $\bar{z}$ the functions $u(z, \bar{z})$ are also analytic functions of two complex variables. In general, they can be continued analytically in the space ranged over by two complex variables and, therefore, outside of their domains of association.

In developing the theory of the functions of $\mathcal{C}(\mathrm{E})$, one may distinguish the following two types of results:

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of partial differential equations of order higher than two, or as solutions of systems of partial differential equations. We note that often the pair of solutions of a system of equations may be interpreted physically, for example, as the stream and potential function of a flow.

It should be stressed that our investigations concern the behavior of functions $u(z, \bar{z})$ for real values of x and y (that is, for z and $\bar{z}$ which are conjugate). However, in some auxiliary considerations we shall extend x and y to complex values.

(4) To every point with the coordinates $x=a$, $y=b$ there correspond planes $\mathfrak{P}^2(a, b) = \mathrm{E}[z=a+ib]$ and $\mathfrak{Q}^2(a, b) = \mathrm{E}[\bar{z}=a-ib]$ in the four-dimensional space. Thus, $\mathfrak{R}^4(\mathfrak{F}^2)$ is the intersection of two four-dimensional cylinders.

(1) Theorems in which $u(z, \bar{z})$ is considered inside of the domain of association, $\mathfrak{A}^2$.

(2) Theorems concerning the behavior of u on the boundary(5) of $\mathfrak{A}^2$, as well as the properties of u outside of $\mathfrak{A}^2$.

Many theorems of the type (1) follow immediately, for the functions u, from corresponding results in the theory of analytic functions, by using the representation(6) (2.6) and the Corollary 3.1 (p. 136). In particular this is true for many theorems stating that an analytic function can be represented as the sum of a linear combination of a finite or infinite number of analytic functions, belonging to a given set. For instance, this is true for theorems dealing with development in series, and on approximation, and the Cauchy integral formula, as well as the many consequences of these theorems. (See §§5 and 6.)

In §7 we show that the connection between the position of certain singularities of $u(z, \bar{z})$ and the coefficients B_{mn} of the development $\sum B_{mn}x^m y^n$ of u is, to a certain degree independent of any special choice of E^* (see (2.7), [2] and [3]). The same holds for various theorems concerning the connection between B_{mn} and the regularity domain, the growth of u and averages of u with certain weight functions. In §9 we study the coincidence, along curves, of the values of functions belonging to two different classes. These considerations show that many properties of the functions u of the class $\mathcal{C}(\mathrm{E})$ are either independent of the choice of E^* (see (2.7)) or depend upon E^* in a simple manner.

In particular since functions u satisfying (1.1) can be presented in the form (2.4) with E_k of the form (2.7) (see [3, §1])(7) these results are valid for the solution of partial differential equations

$$L(U) = 0.$$

Since E_k^* is the only expression in (2.3) which depends on a, b, c, *the resulting relations are independent of the coefficients* a, b, c *of the equation.*

On the other hand, solutions of certain equations $L(U)=0$ form also a class $\mathcal{C}(\mathrm{E})$ wherein E is of a quite different form from that here considered; for instance wherein

(5) The study of singularities of functions of $\mathcal{C}(\mathrm{E})$ is a particular one of this group of questions.

(6) In previous papers [3], [4] we proved that the solutions of an equation $L(u)=0$ can be presented in the form (2.4) with E_k, $k=1, 2$ possessing properties A and B. With this result we constructed two sets of functions $\{\phi_\nu^{(k)}(z, \bar{z})\}$ which serve as bases of the class $\mathcal{C}(\mathrm{E}_k)$ with respect to the star-domains. In [4] we discussed the application of this method to the actual solution of boundary value and characteristic value problems.

(7) The existence of the operator $G(z, \bar{z}, \zeta, \bar{\zeta}, X_{00}, X_{10}, X_{01})$ for these functions follows from Green's formula. (See [11, p. 515, (9)].)

We note that U differs slightly from u. (See (2.4) and (2.5).) This should be kept in mind when formulating results for solutions of differential equations.

$$\mathrm{E} = \left[\sum_{\nu=1}^{m} t^\nu q_\nu(z, \bar{z})\right]\left\{\exp\left[\sum_{\nu=1}^{n} t^\nu p_\nu(z, \bar{z})\right]\right\}. \tag{2.11}$$

(See [3, §3].)

Since various properties of the class $\mathcal{C}(\mathrm{E})$ depend to a large extent upon E, the study of $\mathcal{C}(\mathrm{E})$ with various forms of E gives results which are quite different from the results of this paper(8).

The study of $\mathcal{C}(\mathrm{E})$ with E of the form (2.11) seems to be particularly important for the study of singularities of functions satisfying $L(u)=0$.

It is possible to show that to a pole of the associate function f (of u) there corresponds in this case, a singularity of u with the following property: u satisfies two ordinary differential equations in z and $\bar{z}$, with coefficients which depend in a simple manner upon p_n and q_n. (See [3, §2] and [2].)

3. **Determination of f in terms of $u(z, \bar{z})$, $(z, \bar{z}) \in \mathfrak{R}^4(\mathfrak{F}^2)$.** In this section we shall determine the operator

$$f(\zeta) = \boldsymbol{R}(\zeta \mid u), \tag{3.1}$$

inverse to (2.6). For the sake of simplicity, we shall deal in the future with a certain operator $\boldsymbol{Q}$ instead of $\boldsymbol{R}$. $\boldsymbol{Q}$ is connected with $\boldsymbol{R}$ by the relation(9)

$$\boldsymbol{R}(\xi \mid u) = \frac{(2\xi)^{1/2}}{\Gamma(1/2)} \frac{d^{1/2}\boldsymbol{Q}(2\xi \mid u)}{d\xi^{1/2}}, \tag{3.2}$$

which may be written in the form of an integral relation:

$$\boldsymbol{R}(\xi \mid u) = \frac{2}{\pi}\int_0^{\pi/2} \xi \sin\vartheta \frac{d\boldsymbol{Q}(2\xi \sin^2\vartheta \mid u)}{d(\xi \sin^2\vartheta)} d\vartheta + \pi^{-1}u(0, 0) \tag{3.3}$$

(cf. [3, p. 1177]).

THEOREM 3.1. *We have the relation*

$$\boldsymbol{Q}(\xi \mid u) = u(\xi, 0). \tag{3.4}$$

Proof. It follows from property A (cf. p. 133), that

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(8) As shown in [3], if $\mathrm{E}(z, \bar{z}, t)$ satisfies a certain partial differential equation, then U, $U \in \mathcal{C}(\mathrm{E})$, satisfies the equation $L(U)=0$. Clearly this equation may have many solutions, any of which can be used as E. We note that the equation (1.2) of [3] can be simplified. Introducing $p = tz^{1/2}$ instead of t, equation (1.2) becomes $\mathrm{E}^*_{\bar{z}p} - p^{-1}\mathrm{E}^*_{\bar{z}} + 2p(\mathrm{E}^*_{z\bar{z}} + D\mathrm{E}^*_{z} + F\mathrm{E}) = 0$. E^* is connected with E by the relation (1.8) of [3]. (See also [3, p. 1177].) Finally, writing $\mathrm{E}^* = 1 + pQ(z, \bar{z}, p)$ we obtain $Q_{\bar{z}p} + 2p(Q_{z\bar{z}} + DQ_{\bar{z}} + FQ) + 2F = 0$.

For certain purposes it is also useful to consider an operator of the form $\mathrm{P}(f) = f(z)\mathrm{E}_1(z, \bar{z}) + \int_\alpha^\beta \mathrm{E}_2(z, \bar{z}, s) f(p(z, s))ds$ or, others of more complicated structure.

Note that the above operator transforms $\log z$ into a function with a logarithmic singularity.

(9) We define, as usual, $d^{1/2}(\sum a_n \zeta^n)/d\zeta^{1/2} = \sum (\Gamma(n+1)/\Gamma(n+1/2)) a_n \zeta^{n-1/2}$.

$$u(z, 0) = \int_{-1}^{1} f((z/2)(1 - t^2))(1 - t^2)^{-1/2}dt. \tag{3.5}$$

Suppose now that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$\begin{aligned} u(z, 0) &= \sum_{n=0}^{\infty} a_n (z/2)^n \int_{-1}^{1} (1 - t^2)^{n-1/2}dt \\ &= \sum_{n=0}^{\infty} a_n (z/2)^n \Gamma(n + 1/2)\Gamma(1/2)/\Gamma(n + 1). \end{aligned} \tag{3.6}$$

Since $\boldsymbol{R}(z|u) = f(z) = \sum_{n=0}^{\infty} a_n z^n$, the relation (3.4) follows from (3.2) and (3.6).

If $f(z/2)$ is regular in $\mathfrak{F}^2$, then it follows, by (2.6) and A, that $u(z, \bar{z})$, too, is regular in $\mathfrak{F}^2$. Theorem 3.1 yields the inverse statement given by

COROLLARY 3.1. *Suppose that $u(z, \bar{z})$, $u \in \mathcal{C}(\mathrm{E})$, is regular in a star-domain $\mathfrak{F}^2$. Then $f(z/2)$ is regular in $\mathfrak{F}^2$.*

This fact is an immediate consequence of (3.4). For the regularity of $f(z/2) = \boldsymbol{R}(z/2|u)$ in $\mathfrak{F}^2$ follows by (3.2) from the regularity of $\boldsymbol{Q}(z|u)$ in the same domain. The regularity of $\boldsymbol{Q} \equiv u(z, 0)$ in $\mathfrak{F}^2$ follows from B since the domain $\mathrm{E}[z \in \mathfrak{F}^2, \bar{z} = 0]$ lies in $\mathfrak{M}^4(\mathfrak{F}^2)$.

4. **Determination of the associate function in terms of $u(z, \bar{z})$ in the real plane.** Relations (3.3) and (3.4) give the representation of the associate function. But in this formula there appear functions $u(z, \bar{z})$ for which the values of z and $\bar{z}$ are, in general, not conjugate to each other. This means that we consider $v(x, y) = u(z, \bar{z})$ for complex values of x, y. On the other hand, for many questions it is important to have a formula where $v(x, y)$ appears and takes on only real values of the arguments. We obtain such a formula by substituting the right-hand member of (2.10) for u in (3.4). (See [5, §3].) However, this last formula is inconvenient because the expression obtained for $\boldsymbol{Q}$ depends on G, and therefore on $\mathcal{C}(\mathrm{E})$. Because of the importance of formulas for $\boldsymbol{R}$, we shall indicate other expressions for $\boldsymbol{Q}$ which are independent of $\mathcal{C}(\mathrm{E})$.

THEOREM 4.1. *Suppose that $u(z, \bar{z}) \in \mathcal{C}(\mathrm{E})$ is regular in $\mathrm{E}[x^2 + y^2 \leqq 4R^2]$. Assume $r < R$. Then*

$$\boldsymbol{Q}(\zeta | u) = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \sum_{k=0}^{\infty} (-1)^k \frac{r^k}{k!} \frac{\partial^k}{\partial r^k} \left[\frac{u(re^{i\varphi}, re^{-i\varphi})re^{i\varphi}}{re^{i\varphi} - \zeta} \right] \right\} d\varphi. \tag{4.1}$$

Proof. By (2.7) we have

$$u(z, \bar{z}) = \boldsymbol{Q}(z | u) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} z^m \bar{z}^n, \qquad \boldsymbol{Q}(z | u) = \sum_{m=0}^{\infty} A_{m0} z^m. \tag{4.2}$$

The series (4.2) converges uniformly and absolutely[10] in $\mathrm{E}[|z| < R, |\bar{z}| < R]$.

[10] The absolute convergence follows from $\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} |A_{mn}|\,|z^m\bar{z}^n| \leqq \int_{-1}^{1} [\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} |\mathrm{E}_{mn}(t)| \cdot |z^m\bar{z}^n|\,][\sum_{n=0}^{\infty} |a_n z^n|\,((1-t^2)/2)^n](1-t^2)^{-1/2}dt.$

Integrating along the circle $|z| = r$ of the real plane, we obtain

$$(4.3)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\partial^k}{\partial r^k}\left[\frac{u(re^{i\varphi}, re^{-i\varphi})}{r^q}\right]\right\}\frac{d\varphi}{e^{iq\varphi}} = \frac{\partial^k}{\partial r^k}\left[\sum_{\nu=0}^{\infty} A_{\nu+q,\nu} r^{2\nu}\right].$$

Now $\sum_{\nu=0}^{\infty} A_{\nu+q,\nu}\zeta^{2\nu}$ is an analytic function of the complex variable ζ, $|\zeta| \leqq R$. We introduce a new variable $Z = \zeta - r$, and develop this function about the point $Z = 0$. Then putting $Z = -r$, we obtain

$$(4.4)\qquad A_{q0} = \frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{k=0}^{\infty}(-1)^k\frac{r^k}{k!}\,\frac{\partial^k}{\partial r^k}\left[\frac{u(re^{i\varphi}, re^{-i\varphi})}{r^q}\right]\right\}\frac{d\varphi}{e^{iq\varphi}}.$$

Equation (4.1) follows from (4.4).

REMARK. Analogous considerations yield $Q(z|u)$ in terms of $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$, respectively. In fact, by (4.2) we have

$$u(z, \bar{z}) + \bar{u}(\bar{z}, z) = Q(z\,|\,u) + \overline{Q(z\,|\,u)} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} z^m \bar{z}^n + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \bar{A}_{mn}\bar{z}^m z^n,$$

and

$$(1/2\pi)\int_0^{2\pi}(u + \bar{u})d\varphi = (A_{00} + \bar{A}_{00}) + (A_{11} + \bar{A}_{11})r^2 + (A_{22} + \bar{A}_{22})r^4 + \cdots,$$

$$(1/2\pi)\int_0^{2\pi}(u + \bar{u})e^{-iq\varphi}d\varphi = A_{q0}r^q + (A_{q+1,1} + \bar{A}_{1,q+1})r^{q+2} + (A_{q+2,2} + \bar{A}_{2,q+2})r^{q+4} + \cdots$$

for $q \geqq 1$. Similarly,

$$(1/2\pi)\int_0^{2\pi}(u - \bar{u})d\varphi = (A_{00} - \bar{A}_{00}) + (A_{11} - \bar{A}_{11})r^2 + \cdots.$$

In the same way as before, the associate function

$$(4.5)\qquad f(z) = T(z\,|\,u^{(1)}) + iC(z\,|\,u^{(1)}) = G(z\,|\,u^{(2)}) + iM(z\,|\,u^{(2)})$$

can be determined (to within A_{00}) from either the real or imaginary part of $u = u^{(1)} + iu^{(2)}$.

For some purposes, it is convenient to have a formula for $Q(z|u)$ in which no derivatives of u appear. In order to obtain such an expression, we need certain lemmas.

LEMMA 4.1. *Let* $\mathfrak{r}^1 = \mathrm{E}[0 \leqq r_1 \leqq r \leqq r_2 < R]$. *There exists a set of functions* $\phi_\nu(z)$ *such that*

$$\iint_{|z|<R} \phi_\nu(z)\phi_\mu(z)dxdy = 1, \quad \textit{for} \quad \nu = \mu, \tag{4.6a}$$

$$= 0, \quad \textit{for} \quad \nu \neq \mu,$$

$$\int_{\mathfrak{r}^1} \phi_\nu(r)\bar{\phi}_\mu(r)dr = \lambda_\nu \quad \textit{for} \quad \nu = \mu, \tag{4.6b}$$

$$= 0 \quad \textit{for} \quad \nu \neq \mu.$$

Proof. As is well known, the system $\{(n/\pi)^{1/2}Z^{n-1}\}$ is orthonormal in the unit circle, $|Z| < 1$. Set $h(Z) = \sum a_n(n/\pi)^{1/2}Z^{n-1}$, $g(Z) = \sum b_n(n/\pi)^{1/2}Z^{n-1}$. The Hermitian form

$$\begin{aligned}\int_{\rho_1}^{\rho_2} h(\rho)\bar{g}(\rho)d\rho &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m\bar{b}_n \frac{(nm)^{1/2}}{\pi}\int_{\rho_1}^{\rho_2} \rho^{n+m-2}d\rho \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m\bar{b}_n \frac{(nm)^{1/2}}{\pi(n+m-1)}(\rho_2^{n+m-1} - \rho_1^{n+m-1}), \quad \rho_1 < \rho_2 < 1,\end{aligned} \tag{4.7}$$

is completely continuous (vollstetig), since $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\rho_2^{n+m-1}$ exists (see [9, pp. 147–151, especially p. 151]). Therefore, (4.7) can be written in the form

$$\sum_{s=1}^{\infty} \lambda_s^* \left(\sum_{m=1}^{\infty} \mathfrak{o}_{sm}a_m\right)\left(\overline{\sum_{n=1}^{\infty} \mathfrak{o}_{sn}b_n}\right) \tag{4.8}$$

where $\{\mathfrak{o}_{sn}\}$ is a unitary matrix. Since $\{\mathfrak{o}_{sn}\}$ is unitary, the functions

$$\psi_s(Z) = \sum_{n=1}^{\infty} (n/\pi)^{1/2}\mathfrak{o}_{sn}Z^{n-1} \tag{4.9}$$

have the property that

$$\iint_{|Z|<1} \psi_\nu(Z)\overline{\psi_\mu(Z)}dXdY = 1, \qquad \int_{\rho_1}^{\rho_2} \psi_\nu(\rho)\bar{\psi}_\mu(\rho)d\rho = \lambda_\nu^* \text{ for } \nu = \mu, \tag{4.10}$$

$$= 0, \qquad\qquad = 0 \text{ for } \nu \neq \mu.$$

Substituting in (4.9) $Z = z/R$, $\rho_k = r_k/R$, $\lambda_\nu = \lambda_\nu^*/R$, $\phi_\nu(z) = \psi_\nu(z/R)/R$, we obtain a system of the desired form.

LEMMA 4.2. *Let $g(z)$ be an analytic function of one complex variable z, regular in $|z| \leqq R$, which takes on the values $F(r)$ on $\mathfrak{r}^1$, and for which $\iint_{|z|<R}|g(z)|^2dxdy < \infty$. Then*

$$g(z) = \sum_{\nu=1}^{\infty} \frac{\phi_\nu(z)}{\lambda_\nu}\int_{\mathfrak{r}^1} F(r)\bar{\phi}_\nu(r)dr, \qquad |z| < R. \tag{4.11}$$

Proof. By theorems on orthogonal functions [7 p. 26], it is seen that $g(z)$ can be represented in $|z|<R$ in the form $g(z)=\sum_{\nu=1}^{\infty}A_\nu\phi_\nu(z)$, where the series converges uniformly in $|z|\leqq\rho<R$. Since $\sum_{\nu=1}^{\infty}A_\nu\phi_\nu(r)=F(r)$ for $r\in\mathfrak{r}^1$, the relation

$$\int_{\mathfrak{r}^1}\bar{\phi}_\mu(r)\sum_{\nu=1}^{\infty}A_\nu\phi_\nu(r)dr=\sum_{\nu=1}^{\infty}A_\nu\int_{\mathfrak{r}^1}\phi_\nu(r)\bar{\phi}_\mu(r)dr=\int_{\mathfrak{r}^1}F(r)\bar{\phi}_\mu(r)dr$$

gives us $\lambda_\mu A_\mu=\int_{\mathfrak{r}^1}F(r)\bar{\phi}_\nu(r)dr$, which yields (4.11).

Remark. From (4.11) and (4.9) we have

$$g(0)=R^{-1}\sum_{\nu=1}^{\infty}(\mathfrak{o}_{\nu1}/\lambda_\nu)\int_{\mathfrak{r}^1}F(r)\bar{\phi}_\nu(r)dr. \tag{4.12}$$

Theorem 4.2. *Under hypothesis of Theorem* 4.1, *we have*

$$\mathcal{Q}(\zeta\mid u)=\frac{1}{2\pi R}\left[\sum_{s=1}^{\infty}\int_{\mathfrak{r}^1}\frac{\mathfrak{o}_{s1}\phi_s(r)}{\lambda_s}\int_0^{2\pi}\frac{u(re^{i\varphi},re^{-i\varphi})re^{i\varphi}}{re^{i\varphi}-\zeta}d\varphi dr\right]. \tag{4.13}$$

Proof. By (4.3) (with $k=0$) and (4.12), we have

$$A_{n0}=\frac{1}{2\pi R}\sum_{s=1}^{\infty}\int_{\mathfrak{r}^1}\frac{\mathfrak{o}_{s1}\phi_s(r)}{\lambda_s}\int_0^{2\pi}\frac{u(re^{i\varphi},re^{-i\varphi})}{r^ne^{in\varphi}}d\varphi dr. \tag{4.14}$$

Since $\mathcal{Q}(\zeta|u)=\sum_{n=0}^{\infty}A_{n0}\zeta^n$, the relation (4.13) follows.

Theorem 4.3. *Under the hypotheses of Theorem* 4.1, *we have*

$$\mathcal{Q}(\zeta\mid u)=\frac{1}{2\pi}\sum_{n=0}^{\infty}(n+1/2)P_n(0)\int_{r=-r_1}^{r=r_1}P_n(r/r_1)dr\int_0^{2\pi}\frac{u(re^{i\varphi},re^{-i\varphi})d\varphi}{r_1(1-\zeta/re^{i\varphi})}, \quad |\zeta|<r, \tag{4.15}$$

where the functions P_n are Legendre polynomials.

Proof. Integrating (4.2) multiplied by $e^{-iq\varphi}$ along the circle $|z|=r$, $r\leqq r_1<R$, of the real plane, we obtain

$$\sum_{\nu=0}^{\infty}A_{\nu+q,\nu}r^{2\nu}=\frac{1}{2\pi r^q}\int_0^{2\pi}\frac{u(re^{i\varphi},re^{-i\varphi})}{e^{iq\varphi}}d\varphi. \tag{4.16}$$

Since $\sum_{\nu=0}^{\infty}A_{\nu+q,\nu}r^{2\nu}$ is a function of bounded variation, it can be developed in a uniformly convergent series

$$\begin{aligned}&\sum_{\nu=0}^{\infty}A_{\nu+q,\nu}r^{2\nu}\\&\quad=\sum_{\nu=0}^{\infty}(\nu+1/2)P_\nu(r/r_1)\int_{r=-r_1}^{r=r_1}\frac{dr}{2\pi r^q}P_\nu(r/r_1)\int_0^{2\pi}\frac{u(re^{i\varphi},re^{-i\varphi})d\varphi}{r_1e^{iq\varphi}}.\end{aligned} \tag{4.17}$$

Substituting $r=0$ we obtain (4.15) from (4.17).

II. Duality between the theory of analytic functions of one complex variable and the theory of functions of $\mathcal{C}(\mathrm{E})$

5. **An integral formula for functions of $\mathcal{C}(\mathrm{E})$.** In this section we shall develop an analogue of the Cauchy integral formula.

THEOREM 5.1. *For every point Z there exists a function $H(z, \bar{z}; Z) \in \mathcal{C}(\mathrm{E})$, regular in the region* $\mathrm{E}[|z| < \infty] - \mathrm{E}[z = Zs, 2 \leqq s < \infty]$, *such that every function $u(z, \bar{z}) \in \mathcal{C}(\mathrm{E})$ can be represented by the line-integral*

$$u(z, \bar{z}) = (1/2\pi i) \int_{\mathfrak{a}^1} R(Z \mid u) H(z, \bar{z}; Z) dZ. \tag{5.1}$$

$\mathfrak{a}^1$ is an arbitrary rectifiable closed curve, lying in the domain of association of u, and such that the origin lies in its interior.

Proof. By (2.1) and the Cauchy integral formula we have

$$\begin{aligned} u(z, \bar{z}) &= \int_{-1}^{1} \mathrm{E}(z, \bar{z}, t) f((z/2)(1 - t^2))(1 - t^2)^{-1/2} dt \\ &= (1/2\pi i) \int_{-1}^{1} \mathrm{E}(z, \bar{z}, t)(1 - t^2)^{-1/2} \left[\int_{\mathfrak{a}^1} f(Z)/(Z - (z/2)(1 - t^2)) dZ \right] dt \\ &= (1/2\pi i) \int_{\mathfrak{a}^1} f(Z) \left[\int_{-1}^{1} [\mathrm{E}(z, \bar{z}, t)/(Z - (z/2)(1 - t^2))](1 - t^2)^{-1/2} dt \right] dZ. \end{aligned} \tag{5.2}$$

The expression in the bracket belongs to $\mathcal{C}(\mathrm{E})$; designating it by $H(z, \bar{z}; Z)$, we obtain (5.1) by (3.1).

6. **Development in series and approximation in $\mathcal{C}(\mathrm{E})$.** If $h_n(z)$ converges to a limit function $h(z)$ for $z \in \mathfrak{F}^2$, $n \to \infty$, then by (2.9),

$$\lim_{n \to \infty} [\mathrm{P}(h_n(z))] = \mathrm{P}(h(z)), \qquad z \in \mathfrak{F}^2. \tag{6.1}$$

This fact enables us to prove, in the theory of functions of $\mathcal{C}(\mathrm{E})$, a large group of theorems dealing with normal families, on development in series, and on approximation.

EXAMPLES. I. Suppose a sequence $u_n(z, \bar{z})$, $n = 1, 2, \cdots$ of functions regular in $\mathfrak{F}^2$ is given, with $u_n \in \mathcal{C}(\mathrm{E}_n)$. Let, furthermore, $\lim_{n \to \infty} \mathrm{E}_n(z, \bar{z}, t) = \mathrm{E}(z, \bar{z}, t)$ for $(z, \bar{z}) \in \mathfrak{F}^2$, $-1 \leqq t \leqq 1$. Finally, let $\mathcal{Q}(z|u_n)$ omit (that is, fail to take on) two distinct values. Then $u_n(z, \bar{z})$ form a normal family.

II. As is well known, there exist sets of functions $\{f_\nu(z)\}$ possessing the property that every function f regular in a domain $\mathfrak{F}^2$ can be therein represented in the form $f(z) = \sum a_\nu f_\nu(z)$, where the series converges uniformly in every subdomain $\mathfrak{T}^2$, $\bar{\mathfrak{T}}^2 \subset \mathfrak{F}^2$. To every such theorem corresponds the following analogue: For the domain $\mathfrak{F}^2$ there exists a set of functions $u_n(z, \bar{z})$, $u_n(z, \bar{z}) = \mathrm{P}(f_n(z)) \in \mathcal{C}(\mathrm{E})$, such that every function $u(z, \bar{z}) \in \mathcal{C}(\mathrm{E})$, regular in

$\mathfrak{F}^2$, can be represented in the form $u(z, \bar{z})=\sum_{\nu=1}^{\infty}a_\nu u_\nu(z, \bar{z})$. This series converges (uniformly) in every $\overline{\mathfrak{T}}^2\subset\mathfrak{F}^2$. In the same way, every theorem stating that $f(z)$ can be approximated by $\sum_{\nu=1}^{n}a_\nu^{(n)}f_\nu(z)$ in every subdomain $\mathfrak{T}^2$ of $\mathfrak{F}^2$, has an analogue which can be proved in the theory of $\mathcal{C}(\mathrm{E})$. We note that in certain cases, it is possible to approximate $u(z, \bar{z})$ in $\overline{\mathfrak{F}}^2$ (cf. [4]).

The set $\{z^\nu\}$ plays an important role among the sets of functions f_ν mentioned above. There arises the problem of characterizing the functions $\mathrm{P}(z^{\nu-1})$ independently of their integral representation. This is, in fact, possible if E satisfies a certain differential equation. For then the $\mathrm{P}(z^{\nu-1})$ satisfy an ordinary differential equation. (We shall consider this question in another paper.) In particular, the previous results yield: Every function $u(z, \bar{z})\in\mathcal{C}(\mathrm{E})$ can be developed in every circle of regularity, $|z|<\rho$, in the form $\sum_{\nu=1}^{\infty}a_\nu\mathrm{P}(z^{\nu-1})$ and it can be approximated by $\sum_{\nu=1}^{n}a_\nu^{(n)}\mathrm{P}(z^{\nu-1})$ in every regularity domain $\mathfrak{F}^2$.

In addition, our method enables us to prove immediately many other theorems concerning the degree of approximation. For instance: Let $w=d(z)$ map conformally the complement of $\mathfrak{F}^2$ into $|w|>1$, and denote by $\mathfrak{C}_R^1$ the curve $d(z)=R>1$. If $u(z, \bar{z})$ is analytic in a domain $\mathfrak{F}^2\supset\mathfrak{C}_R^1$, then there exist expressions $p_n(z, z)=\sum_{\nu=1}^{n}a_\nu^{(n)}\mathrm{P}(z^{\nu-1})$ such that

$$\limsup_{n\to\infty}\left[\max\,|u(z, \bar{z}) - p_n(z, \bar{z})|^{1/n}\right] = 1/R.$$

This result is an immediate generalization of the corresponding theorem of Walsh [12].

7. **Coefficient problems.** In an analogous way other results (for instance, those on overconvergence, on existence of boundary values, various gap theorems, and so on) can be proved in the theory of functions of the class $\mathcal{C}(\mathrm{E})$. In §6, we introduced the system $\mathrm{P}(z^{\nu-1})\nu=1, 2, \cdots$. We indicated that the series

$$\sum_{\nu=1}^{\infty} a_\nu\mathrm{P}(z^{\nu-1}) \tag{7.1}$$

has a behavior analogous to that of a power series in the case of analytic functions of one complex variable. In particular, one can deduce various properties of $u(z, \bar{z})$ from the behavior of the coefficients a_ν of its expansion (7.1). On the other hand, the function $u(z, \bar{z})$ can be represented in the neighborhood of the origin in the form

$$\sum_{\nu=0}^{\infty}\sum_{\mu=1}^{\infty} A_{\nu\mu}z^\nu\bar{z}^\mu, \quad\text{or}\quad \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} B_{\nu\mu}x^\nu y^\mu, \tag{7.2}$$

the series converging in $\mathrm{E}[|z|<\rho,\ |\bar{z}|<\rho]$, ρ sufficiently small. The problem now arises of finding properties of u from the behavior of $A_{\nu\mu}$ or $B_{\nu\mu}$.

By (3.4) we have the relation

$$(7.3) \qquad a_m = c_m A_{m0}, \qquad c_m = 2^m \Gamma(m+1)/\pi^{1/2}\Gamma(m+1/2)$$

for the coefficient a_m of the function f, which is the associate of u. Thus, if some property of A_{m0} is known, the corresponding property of a_m follows by (7.3). Then, using the theorems of the theory of analytic functions of one complex variable, which deal with the relation between the function and the coefficients of its series development, we may obtain results concerning the relation between the function $u(z, \bar{z})$ and the coefficients A_{m0} of its development (7.2).

EXAMPLES. I. The radius, r, of the largest circle with center at the origin, inside which $u(z, \bar{z}) = \sum A_{mn} z^m \bar{z}^n$ is regular, is given by

$$(7.4) \qquad 1/r = \lim_{n\to\infty} (|A_{n0}| c_n)^{1/n}.$$

II. Suppose now that $A_{n0} = 0$ in (7.2) for all n, except for $n = \lambda_\nu$, $\nu = 1, 2, \cdots$, where $\lambda_{\nu+1} - \lambda_\nu > \lambda_\nu \theta$, $\theta > 1$. Then $u(z, \bar{z})$ cannot be continued analytically to the outside of the circle whose radius is given by (7.4).

III. A classical result of the theory of entire functions states: Let $f(z) = \sum a_n z^n$ be an entire function. The logarithm of the greatest of the terms $|a_n r^n|$ is asymptotically equal to $\log[\max_{0\leq\varphi\leq 2\pi} |f(re^{i\varphi})|]$. A similar result is valid in the case of entire functions $u(z, \bar{z}) \in \mathcal{C}(\mathrm{E})$. Namely, we have the inequality

$$(7.5) \qquad |u(re^{i\varphi}, re^{-i\varphi})| \leq |A_{n0}R^n|_{\max} \cdot \mathrm{E}_{\max}(r) \cdot (R/R - r),$$

where

$$r < R, \quad |A_{n0}R^n|_{\max} = \max_{n=0,1,2,\cdots} |A_{n0}R^n|, \quad \mathrm{E}_{\max}(r) = \max_{|z|\leq r; -1\leq t\leq 1} |\mathrm{E}(z, \bar{z}, t)|.$$

For we have

$$|u(z, \bar{z})| \leq \int_{-1}^{+1} |\mathrm{E}(z, \bar{z}, t)| \cdot \sum_{n=0}^{\infty} |a_n z^n| (1/2)^n (1-t^2)^n (1-t^2)^{-1/2} dt$$

$$\leq \mathrm{E}_{\max}(r) \cdot \sum_{n=0}^{\infty} |A_{n0} r^n| \leq \mathrm{E}_{\max}(r) \cdot |A_{n0}R^n|_{\max} \cdot (R/R - r).$$

An inequality for A_{n0} in terms of $\max_{0\leq\varphi\leq 2\pi} |u(re^{i\varphi}, re^{-i\varphi})|$ follows from (4.14).

The relation (7.3) enables us to give interesting formulations of many theorems which have analogues in the theory of functions of $\mathcal{C}(\mathrm{E})$. For instance a generalization of a theorem of Fatou type was given in [5](11). Since the coefficients of the associate function can be expressed by A_{n0} in the form (7.3), it follows from [5]: If $\sum_{m=0}^{\infty} |A_{m0}|^2 < \infty$, then $u = \sum A_{mn} z^m \bar{z}^n = \mathrm{P}(f)$, $u \in \mathcal{C}(\mathrm{E})$, possesses boundary values almost everywhere on the unit

(11) We note that on p. 668, in 1.14 of [5] it is necessary to add to $1 + z\bar{z}t^2\mathrm{E}^*$ the factor $\exp(-\int_0^{\bar{z}} A\, d\bar{z})$.

circle. Further, the set of points in which these boundary values exist, includes the set $\mathfrak{C}$, $\mathfrak{C}$ being the set of points in which $\int_{-1}^{1} f((z/2)(1-t^2))(1-t^2)^{-1/2}dt$ has boundary values.

A generalization of Hadamard's multiplication theorem was given in [2].

For certain applications it is useful to obtain an expression for A_{mn} in terms of A_{m0}. Such a formula follows from the fact that

$$u(z, \bar{z}) = u(z_1 + iz_2, z_1 - iz_2)$$

is an analytic function of two complex variables. Thus, we have

$$\begin{aligned} A_{mn} = &-\frac{1}{4\pi^2 n! m!}\int_0^{2\pi}\int_0^{2\pi}\int_{-1}^{1}\{\mathrm{E}[r_1e^{i\varphi_1} + ir_2e^{i\varphi_2},\ r_1e^{i\varphi_1} - r_2e^{i\varphi_2},\ t] \\ &\times \sum_{n=0}^{\infty} A_{n0}[(1/2)(r_1e^{i\varphi_1} + ir_2e^{i\varphi_2})(1-t^2)]^n / r_1^m r_2^n e^{im\varphi_1} e^{in\varphi_1}(1-t^2)^{1/2}\}\, d\varphi_1 d\varphi_2 dt. \end{aligned} \tag{7.6}$$

Using the general integral formula (see [6]), we obtain integral formulas with various ranges of integration.

III. Some problems involving functions of class $\mathcal{C}(\mathrm{E})$

8. **Conjugate functions. Mapping by the functions of $\mathcal{C}(\mathrm{E})$.** The functions u considered in this paper are complex. In most applications (theory of linear partial differential equations) we need consider only their real part. We wish to indicate a problem which involves both the real and imaginary part of u.

The equation (1.1) is equivalent to the system of two equations

$$\begin{aligned} &(1/4)\Delta U^{(1)} + (1/2)AU_x^{(1)} + (1/2)BU_y^{(1)} + (1/2)CU_x^{(2)} + (1/2)DU_y^{(2)} \\ &\qquad + c_1U^{(1)} - c_2U^{(2)} = 0, \\ &(1/4)\Delta U^{(2)} - (1/2)CU_x^{(1)} - (1/2)DU_y^{(1)} + (1/2)AU_x^{(2)} + (1/2)BU_y^{(2)} \\ &\qquad + c_2U^{(1)} + c_1U^{(2)} = 0 \end{aligned} \tag{8.1}$$

where

$$U = U^{(1)} + iU^{(2)}, \qquad c = c_1 + ic_2, \qquad A = (1/2)[(a + \bar{a}) + (b + \bar{b})],$$
$$B = (1/2i)[(a - \bar{a}) - (b - \bar{b})], \qquad C = (-1/2i)[(a - \bar{a}) + (b - \bar{b})],$$
$$D = (1/2)[(a + \bar{a}) - (b + \bar{b})].$$

On the other hand, every solution of (1.1) can be written in the form, $\exp(-\int_0^{\bar{z}} a d\bar{z}) \cdot u(z, \bar{z}) + \exp(-\int_0^z b dz) \cdot v(z, \bar{z})$, where $v(z, \bar{z})$ belongs to a class of functions, whose associates are anti-analytic functions (that is, analytic functions of $\bar{z}$). Thus the functions of $\mathcal{C}(\mathrm{E})$, with an appropriate E, form a subclass of the functions satisfying the system (8.1). However, if $a = \bar{b}$, and c is real, then the equations (8.1) are independent of each other.

Relations existing between the real and imaginary parts of u, in the general case, are given in

THEOREM 8.1. *Let* $u=u^{(1)}+iu^{(2)}\in\mathcal{C}(\mathrm{E})$, $\mathrm{E}=\mathrm{E}^{(1)}+i\mathrm{E}^{(2)}$. *Further, let* $T(z|u^{(1)})+iC(z|u^{(1)})$ *be the associate of* u (*see* (4.5)). *Then*

$$
\begin{aligned}
u_x^{(1)} - u_y^{(2)} &= \int_{-1}^{1} (\mathrm{E}_x^{(1)} - \mathrm{E}_y^{(2)}) T(z \mid u^{(1)})(1-t^2)^{-1/2} dt \\
&\quad - \int_{-1}^{1} (\mathrm{E}_x^{(2)} + \mathrm{E}_y^{(1)}) C(z \mid u^{(1)})(1-t^2)^{-1/2} dt,
\end{aligned}
\tag{8.2}
$$

$$
\begin{aligned}
u_x^{(2)} + u_y^{(1)} &= \int_{-1}^{1} (\mathrm{E}_x^{(2)} + \mathrm{E}_y^{(1)}) T(z \mid u^{(1)})(1-t^2)^{-1/2} dt \\
&\quad + \int_{-1}^{1} (\mathrm{E}_x^{(1)} - \mathrm{E}_y^{(2)}) C(z \mid u^{(1)})(1-t^2)^{-1/2} dt.
\end{aligned}
\tag{8.3}
$$

We obtain (8.2) and (8.3) by differentiating u and using the Cauchy-Riemann differential equations for the associate.

In addition to $u^{(1)}$ and $u^{(2)}$ we may consider the pair of functions $v^{(1)}$, $v^{(2)}$, where $v^{(1)}+iv^{(2)}=\int_{-1}^{1}\mathrm{E}_1(z,\bar{z},t)f[(z/2)(1-t^2)](1-t^2)^{-1/2}dt$ and

$$\mathrm{E}_1=(\mathrm{E}_x^{(1)}-\mathrm{E}_y^{(2)})+i(\mathrm{E}_x^{(2)}+\mathrm{E}_y^{(1)}).$$

The functions $u^{(k)}$ and $v^{(k)}$, $k=1, 2$, are connected by the equations

$$u_x^{(1)} - u_y^{(2)} = v^{(1)}, \qquad u_y^{(1)} + u_x^{(2)} = v^{(2)}. \tag{8.4}$$

It follows that many relations exist between $u^{(k)}$ and $v^{(k)}$. For instance, if $u^{(1)}$ satisfies a (self-adjoint) partial differential equation of elliptic type and second order, say $L_1(u^{(1)})=\Delta u^{(1)}+4cu^{(1)}=0$, c real, then a generalized Cauchy formula is valid. It determines the values of $u^{(1)}$ inside a domain $\mathfrak{A}^2$ in terms of the values of $u^{(k)}$ and $v^{(k)}$, $k=1, 2$, on the boundary $\mathfrak{a}^1$ of $\mathfrak{A}^2$. Using the formula (9), p. 515 of [11] we obtain

$$
\begin{aligned}
2\pi u^{(1)}(x, y) = \int_{\mathfrak{a}^1} &[(u^{(2)}W_\xi - u^{(1)}W_\eta + v^{(2)}W)d\xi \\
&+ (u^{(2)}W_\eta + u^{(1)}W_\xi - v^{(1)}W)d\eta], \qquad (x, y) \in \mathfrak{A}^2.
\end{aligned}
$$

Here $W(x, y; \xi, \eta)$ is a fundamental solution of L_1.

REMARK. Clearly, if $u^{(k)}$ are connected by the generalized Cauchy-Riemann equations,

$$\sum_{k=1}^{2} [\overset{(s)}{a}_{1k} \overset{(k)}{u}_x + \overset{(s)}{a}_{2k} \overset{(k)}{u}_y + \overset{(s)}{a}_{3k} \overset{(k)}{u} + \overset{(s)}{a}_{4k}] = 0, \qquad s = 1, 2,$$

$$\overset{(1)}{a}_{s1} \overset{(2)}{a}_{s2} - \overset{(2)}{a}_{s1} \overset{(1)}{a}_{s2} \neq 0,$$

a generalized Cauchy formula can be obtained without introducing $v^{(1)}$, $v^{(2)}$. If $L_1(u^{(1)})=0$ then

$$2\pi u^{(1)}(x, y) = \int_{\mathfrak{a}^1} \{ [(-(A_2W)_\xi + WC_2 - W_\eta)u^{(1)} + (-(B_2W)_\xi + D_2W)u^{(2)} + E_2W]d\xi - [(-(WA_1)_\eta + WC_1 - W_\xi)u^{(1)} + (-(WB_1)_\eta + WD_1)u^{(2)} + WE_1]d\eta \}, \qquad (x, y) \in \mathfrak{A}^2.$$

A_k, B_k, $\cdots$ are polynomials in $a_{sn}^{(k)}$, they are the coefficients of the expressions

$$u_\xi^{(1)} = A_1 u_\eta^{(1)} + B_1 u_\eta^{(2)} + C_1 u^{(1)} + D_1 u^{(2)} + E_1,$$
$$u_\eta^{(l)} = A_2 u_\xi^{(1)} + B_2 u_\xi^{(2)} + \cdots .$$

In analogy with conformal transformations, one may consider the mapping of the (x,y)-plane by the functions $U(z,\bar{z})$ of the class $\exp(-\int_0^{\bar{z}} a d\bar{z})\cdot \mathcal{C}(\mathrm{E})$. If U satisfies (1.1) then this mapping represents a transformation by a pair $(U^{(1)}, U^{(2)})$ of solutions of the system (8.1). The following case is of special interest. Suppose that the boundary $\mathfrak{f}^1$ of $\mathfrak{F}^2$ can be decomposed: $\mathfrak{f}^1 = \mathfrak{S}_{\nu=1}^n \mathfrak{f}_\nu^1$. Suppose further, that by the transformation $U=U(z, \bar{z})$ every curve-segment $\mathfrak{f}_\nu^1$ is transformed into $\mathfrak{g}_\nu^1 = \mathrm{E}[\psi_\nu(U^{(1)}, U^{(2)})=0]$, $\nu=1, 2, \cdots, n$. The pair $(U^{(1)}, U^{(2)})$ then represents a solution of the system (8.2), (8.3), satisfying the boundary condition

$$\psi_\nu(U^{(1)}, U^{(2)}) = 0 \text{ on } \mathfrak{f}_\nu^1, \qquad \nu = 1, 2, \cdots, n.$$

9. **The coincidence of functions of different classes along curves.** In this section we investigate the problem: when can two functions of different classes, or at least their real parts, coincide along a curve. Results in this direction are especially of interest if one of the two classes is the class of analytic functions of one complex variable.

We shall indicate some applications of the results in this direction to the boundary value problem(12), and to the characterization of singularities.

Let $\mathrm{E}(z, \bar{z}, t)$ *possess the property that for* (x, y), $z=x+iy$, $\bar{z}=x-iy$, *belonging to a curve* $\mathfrak{k}^1$ *of the real plane, we have*

$$\mathrm{E}(z, \bar{z}, t) = \mathrm{E}_1(z, t), \qquad (x, y) \in \mathfrak{k}^1, \tag{9.1}$$

(12) In analogy with the theory of partial differential equations, we may consider the boundary value problems for the functions of the class $\mathcal{C}(\mathrm{E})$. Since a function which satisfies (1.1) can be represented in the form (2.4), the boundary value problem for $L(U)=0$ can be reduced to that of functions of the class $\mathcal{C}(\mathrm{E})$. We note that if $a=\bar{b}$ and c is real (see (1.1)), we may write $U(z, \bar{z}) = \mathrm{Re}\{\exp(-\int_0^{\bar{z}} a d\bar{z})\int_{-1}^1 \mathrm{E}(z, \bar{z}, t) f((z/2)(1-t^2))(1-t^2)^{-1/2}dt\}$.

In this case our later results can be directly used in the theory of partial differential equations.

where $E_1(z, t)$ *is an analytic function of one complex variable regular in a (sufficiently large) domain of the (complex) z-plane. Then*

$$u(z, \bar{z}) = h(z) \qquad \textit{for } (x, y) \in \mathfrak{k}^1 \tag{9.2}$$

where $u(z, \bar{z})$ *is the function of* $\mathcal{C}(E)$ *introduced by* (2.6) *and*

$$h(z) = \int_{-1}^{1} E_1(z, t) f\left(z \frac{1 - t^2}{2}\right) \frac{dt}{(1 - t^2)^{1/2}} \tag{9.3}$$

is an analytic function of one complex variable.

EXAMPLE I. We have

$$E(z, \bar{z}, t) = E(z, 2ic + z, t) \equiv E_1(z, t) \quad \text{for } (x, y) \in \mathfrak{k}_1^1 = E[y = -c]. \tag{9.4}$$

EXAMPLE II. Suppose that

$$E(z, \bar{z}, t) = E_1(r, t) = 1 + t^2 E_1^*(r, t), \qquad r^2 = x^2 + y^2, \tag{9.5}$$

where $E_1(r, t)$ is a function which is independent of φ.

(See also (2.3).) Here r and φ are polar coordinates. Then we have

$$E(z, \bar{z}, t) = E_1(\rho, t) \qquad \text{for } (x, y) \in \mathfrak{k}_2^1 = E[x^2 + y^2 = \rho^2]. \tag{9.6}$$

We now shall discuss the above mentioned applications of the coincidence of the functions u and h on $\mathfrak{k}^1$ (see (9.2)).

1. *Boundary value problem.* We consider at first the case where E is of the form described in Example II. Let $u(z, \bar{z}) \in \mathcal{C}(E)$, where E satisfies (9.6). If for all integers n, $n \geqq 0$,

$$J_n(\rho) \equiv \int_{-1}^{1} E_1(\rho, t)(1 - t^2)^{n-1/2} dt \neq 0,$$

then $\tilde{u}(r, \varphi) \equiv u(re^{i\varphi}, re^{-i\varphi})$ can be represented in the domain $\mathfrak{R}^2 = E[|z| < \rho]$ in the form

$$\begin{aligned} \tilde{u}(r, \varphi) &= \frac{1}{2\pi} \int_0^{2\pi} v(\vartheta) \sum_{n=0}^{\infty} \frac{r^n J_n(r) e^{in(\varphi - \vartheta)} d\vartheta}{\rho^n J_n(\rho)} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1} v(\vartheta) E_1(r, \tau) \sum_{n=0}^{\infty} \frac{r^n (1 - \tau^2)^{n-1/2} e^{in(\varphi - \vartheta)} d\vartheta d\tau}{\rho^n J_n(\rho)} \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{-1}^{1} \int_0^{2\pi} \frac{v(\vartheta) E_1(r, \tau) H(be^{-i\alpha})}{1 - rb^{-1}\rho^{-1}(1 - \tau^2) e^{i(\alpha + \varphi - \vartheta)}} \frac{d\tau d\alpha d\vartheta}{(1 - \tau^2)^{1/2}}, \end{aligned} \tag{9.7}$$

where $v(\vartheta) = \tilde{u}(\rho, \vartheta)$ is supposed to be an absolutely integrable function, $r/\rho < b < 1$, and

$$H(be^{-i\alpha}) = \sum_{n=0}^{\infty} \frac{b^n e^{-in\alpha}}{J_n(\rho)} \cdot \tag{9.8}$$

We obtain (9.7) by formal calculation, since it follows from Lemma 9.1 that the first series converges uniformly for $r \leqq \rho_0 < \rho$.

LEMMA 9.1. *For every* ϵ, $\epsilon > 0$, *and every* ρ, $\rho \leqq \rho_1 < \infty$ *there exists an* n_0, *such that for* $n > n_0$ *we have*

$$(9.9)\qquad \begin{gathered} 1 - \epsilon \leqq \left| \int_{-1}^{1} \mathrm{E}_1(\rho, t)(1 - t^2)^{n-1/2} dt \Big/ \int_{-1}^{1} (1 - t^2)^{n-1/2} dt \right| \leqq 1 + \epsilon, \\ \int_{-1}^{1} (1 - t^2)^{n-1/2} dt = \frac{\pi^{1/2}\Gamma(n + 1/2)}{\Gamma(n + 1)} . \end{gathered}$$

Proof. By (9.5) we have

$$\begin{aligned} \int_{-1}^{1} \mathrm{E}_1(\rho, t)(1 - t^2)^{n-1/2} dt &= \int_{-1}^{1} (1 - t^2)^{n-1/2} dt + \int_{-1}^{+1} t^2 \mathrm{E}_1^*(\rho, t)(1 - t^2)^{n-1/2} dt \\ &= \frac{\Gamma(1/2)\Gamma(n + 1/2)}{\Gamma(n + 1)} + \int_{-1}^{1} t^2 \mathrm{E}_1^*(\rho, t)(1 - t^2)^{n-1/2} dt. \end{aligned}$$

(See [10, p. 133 formula (2)].) Since $\mathrm{E}_1^*(\rho, t)$ is supposed to be regular, there exists a constant c, such that $|\mathrm{E}_1^*(\rho, t)| \leqq c$, and therefore

$$\begin{aligned} \left| \int_{-1}^{1} t^2 \mathrm{E}_1^*(\rho, t)(1 - t^2)^{n-1/2} dt \right| &\leqq c \int_{-1}^{+1} t^2 (1 - t^2)^{n-1/2} dt \\ &= \frac{c}{2} \frac{\Gamma(1/2)\Gamma(n + 1/2)}{\Gamma(n + 2)}. \end{aligned}$$

Hence from $\Gamma(n+2) = (n+1)\Gamma(n+1)$ we have

$$\begin{aligned} 1 - \frac{c}{2(n + 1)} &\leqq \left| \int_{-1}^{1} \mathrm{E}_1(\rho, t)(1 - t^2)^{n-1/2} dt \Big/ \frac{\Gamma(1/2)\Gamma(n + 1/2)}{\Gamma(n + 1)} \right| \\ &\leqq 1 + \frac{c}{2(n + 1)} \end{aligned}$$

which yields (9.9).

Since in (9.7) we supposed $r \leqq \rho_0 < \rho$ the absolute and uniform convergence of the series in the first integral of (9.7) follows from (9.9).

An analogous formula can be derived if the derivative $\partial \tilde{u}(r, \varphi)/\partial r$ is prescribed along $\mathfrak{k}_2^1$.

REMARK 1. If a (real) function satisfying $L(U) = 0$ and assuming given real values on the boundary has to be determined, we take for $v(\varphi)$ such an analytic function of one complex variable that $\mathrm{Re}[\exp(-\int_0^{\bar{z}} a d\bar{z}) \cdot v(z)]$ assumes the given values of $\mathfrak{k}_2^1$.

REMARK 2. In our above considerations, the existence of functions $u(z, \bar{z}) \in \mathcal{C}(\mathrm{E})$ satisfying the required boundary conditions was presupposed. However, there exist cases where the proof can be given without a preliminary existence hypothesis[13]. For instance, suppose that a (real) function $H(\varphi)$ is chosen in such a way that $H(\varphi)=\operatorname{Re}[h(\rho e^{i\varphi})]$, where $h(z)$ is an analytic function of one complex variable such that $\int_0^{2\pi}|h(\lambda e^{i\varphi})|^p d\varphi<\infty$, $p>1$.

Let $\sum(a_n\cos n\varphi+b_n\sin n\varphi)$ be the Fourier development of $H(\varphi)$. Then

$$\sum c_n r^n\rho^{-n}e^{in\varphi}, \qquad c_n=a_n-ib_n,$$

will be an analytic function, regular in $\mathfrak{K}^2=\mathrm{E}[|z|<\rho]$ the real part of which converges almost everywhere to $H(\varphi)$ when we approach $\mathfrak{k}_2^1$ radially. It follows by Lemma 9.1 that

$$\sum c_n r^n J_n(r)e^{in\varphi}/\rho^n J_n(\rho) \tag{9.10}$$

converges uniformly for every $r\leqq\rho_0<\rho$. Thus (9.10) is a function of $\mathfrak{C}(\mathrm{E})$ which is regular in $\mathfrak{K}^2$, and it suffices to show that it converges to $h(\rho e^{i\varphi})$ as $r\to\rho$. We shall show that

$$\left|\left[\sum_{n=n_0}^{\infty}c_n\frac{r^n}{\rho^n}e^{in\varphi}-\sum_{n=n_0}^{\infty}c_n\frac{r^nJ_n(r)}{\rho^nJ_n(\rho)}e^{in\varphi}\right]\right| \tag{9.11}$$

converges (uniformly in r) to zero, as $n_0\to\infty$. We have

$$\begin{aligned}&\left|\sum_{n=n_0}^{m}c_n\frac{r^n}{\rho^n}e^{in\varphi}-\sum_{n=n_0}^{m}c_n\frac{r^nJ_n(r)}{\rho^nJ_n(\rho)}e^{in\varphi}\right|\\
&\leqq\sum_{n=\nu_0}^{m}\left|c_n\frac{r^n}{\rho^n}\right|\cdot\left|\frac{J_n(\rho)-J_n(r)}{J_n(\rho)}\right|\\
&\leqq\sum_{n=n_0}^{m}\left|c_n\frac{r^n}{\rho_n}\right|\cdot\left|\int_{-1}^{1}[\mathrm{E}_1(\rho,t)-\mathrm{E}_1(r,t)](1-t^2)^{n-1/2}dt\right|\Big/\int_{-1}^{+1}\mathrm{E}_1(\rho,t)(1-t^2)^{n-1/2}dt.\end{aligned}$$

Since $\mathrm{E}_1(\rho,t)$ is an analytic function of the real variable ρ, it satisfies, for all r, a Lipschitz condition

$$|\mathrm{E}_1(\rho,t)-\mathrm{E}_1(r,t)|\leqq C_1|\rho-r|$$

where C_1 is a fixed constant.

Thus (9.11) is smaller than

$$\sum_{n=n_0}^{m}C_1|\rho-r|\left[\int_{-1}^{1}(1-t^2)^{n-1/2}dt\Big/\int_{-1}^{1}\mathrm{E}_1(\rho,t)(1-t^2)^{n-1/2}dt\right]\cdot|c_n|\frac{r^n}{\rho^n}\cdot$$

---

[13] In this case we thus obtain the proof of the existence of a function $u(z,\bar z)\in\mathfrak{C}(\mathrm{E})$ regular in $\mathrm{E}[|z|<\rho]$, the real part of which assumes the prescribed values almost everywhere on the boundary $\mathrm{E}[|z|=\rho]$. (We therefore obtain, in certain instances, an existence proof for solutions of partial differential equations.)

By a known result $\sum_{n=n_0}^{m}|c_n|r^n\rho^{-n}\leqq\epsilon_1(n_0)/(\rho-r)$ where $\lim_{n\to\infty}\epsilon_1(n)=0$. (See [8, pp. 405–408].) Thus, by (9.9), (9.11) is less than

$$\epsilon_1(n_0)[1+\epsilon(n_0)]=\epsilon_2(n_0),\qquad \lim_{n\to\infty}\epsilon_2(n)=0,$$

This completes the proof.

In the last paragraph of §8, we considered, a function $u(z,\bar{z})=u^{(1)}+iu^{(2)}$ which maps a domain $\mathfrak{F}^2+\mathfrak{f}^1$ into the domain $\mathfrak{G}^2+\mathfrak{g}^1$, ($\mathfrak{f}^1=\mathfrak{S}_{\nu=1}^{n}\mathfrak{f}_\nu^1$, $\mathfrak{g}^1=\mathfrak{S}_{\nu=1}^{n}\mathfrak{g}_\nu^1$). The functions $u^{(1)}$ and $u^{(2)}$ are solutions of a system of partial differential equations for which the boundary conditions are: $\psi_\nu[u^{(1)},u^{(2)}]=0$ on $\mathfrak{f}_\nu^1$, $\nu=1,2,\cdots,n$.

If $\mathfrak{F}^2=\mathfrak{K}_2^2=\mathrm{E}[|z|<\rho]$ and if $\psi_\nu(u^{(1)},u^{(2)})$ are linear functions of $u^{(1)}$ and $u^{(2)}$, the solution of the above boundary value problem for a pair of harmonic functions can be written (in special cases)(14) in the form $u^{(1)}+iu^{(2)}=c\int_0^z\prod_{\nu=1}^{n}(z-a_\nu)^{\mu_\nu}dz$. This result can be generalized for the functions $u(z,\bar{z})\in\mathcal{C}(\mathrm{E})$, E satisfying (9.5). In fact, since by (9.2) $h(z)$ and $u(z,\bar{z})$ coincide on $\mathfrak{k}_2^1$, the determination of $u(z,\bar{z})$ can be reduced to the finding of a function $f(z)$ which satisfies the integral equation

$$\int_{-1}^{1}\mathrm{E}_1(\rho,t)f\left(z\frac{1-t^2}{2}\right)\frac{dt}{(1-t^2)^{1/2}}=h(z),\qquad z\in\mathfrak{K}_2^2, \tag{9.12}$$

$h(z)$ being the analytic function satisfying the prescribed boundary conditions. We can develop $f(z)$ and $h(z)$ in power series in the domain $\mathfrak{K}_2^2$. Writing $h(z)=\sum_{n=0}^{\infty}a_nz^n$ and comparing coefficients, we get

$$f(\zeta)=\sum_{n=0}^{\infty}\frac{2^na_n\zeta^n}{J_n(\rho)}=\frac{1}{2\pi}\int_0^{2\pi}h\left(\frac{2\zeta}{b}e^{i\alpha}\right)\mathrm{H}(be^{-i\alpha})d\alpha, \tag{9.13}$$

where H is the function introduced in (9.8), and $|\zeta|<b<1$. Since in the case considered we have $h(z)=\int_0^z\prod_{\nu=1}^{n}(z-a_\nu)^{\mu_\nu}dz$, we have

$$\begin{aligned} u(z,\bar{z})&=\frac{1}{2\pi}\int_{-1}^{1}\int_0^{2\pi}\int_0^{s}\mathrm{E}(z,\bar{z},t)\mathrm{H}(be^{-i\alpha})\\ &\cdot\left[\prod_{\nu=1}^{n}(\eta-a_\nu)^{\mu_\nu}\right](1-t^2)^{-1/2}d\eta d\alpha dt,\ s=z(1-t^2)e^{i\alpha}/b. \end{aligned} \tag{9.14}$$

---

(14) We note that in the case considered $u^{(1)}$ and $u^{(2)}$ satisfy the Cauchy-Riemann equations in addition to the potential equation.

Since $u^{(1)}+iu^{(2)}=\exp(\int_0^{\bar{z}}ad\bar{z})U$ (see (2.5)) the relations $A_\nu u^{(1)}+B_\nu u^{(2)}+C_\nu=0$, $\nu=1,2,\cdots,n$; A_ν,B_ν,C_ν being constants, become $(A_\nu+B_\nu)p_1U^{(1)}+(A_\nu-B_\nu)p_2U^{(2)}+C_\nu=0$ where p_1 and p_2 are the real and imaginary parts, respectively, of $\exp(\int_0^{\bar{z}}ad\bar{z})$.

We remark that when dealing with differential equations, especially in connection with the coincidence problem, it is often useful to consider classes $\mathcal{C}(\mathrm{E})$ with a generating function $\mathbf{E}$ which does not fulfill the hypothesis A (see p. 133).

In the case I (see p. 146) we can proceed similarly. However, the determination of f from (9.3) is slightly more complicated. Let $h(z)$ be the analytic function, regular in $\mathfrak{R}_1^2 = E[y > -c]$, $c > 0$, which assumes the given values on the curve $\mathfrak{k}_1^1 = E[y = -c]$. Since it is a convex domain containing the origin, there exists, by Corollary 3.1, an analytic function $f(z) = \boldsymbol{R}(z|u)$ (see (3.1)) such that

$$\text{(9.15)} \qquad \int_{-1}^{1} \mathrm{E}(z, z + 2ic, t) f\left(z \frac{1 - t^2}{2}\right) \frac{dt}{(1 - t^2)^{1/2}} = h(z).$$

Let $\sum \alpha_n z^n$ and $\sum A_n z^n$ be the function elements of f and h, respectively, at the origin and suppose $\mathrm{E}(z, z+2ic, t) = \sum \mathrm{P}_n(t) z^n$. Then we have

$$\text{(9.16)} \qquad \alpha_n = \begin{vmatrix} E_{0,0} & 0 & 0 & \cdots A_0 \\ E_{1,0} & E_{0,1} & 0 & \cdots A_1 \\ \cdot & \cdot & \cdot & \cdot \\ E_{n,0} & E_{n-1,1} & E_{n-2,2} & \cdots A_n \end{vmatrix} \Bigg/ \prod_{\nu=0}^{n} E_{0,\nu},$$

$$E_{n,k} = \int_{-1}^{1} \mathrm{H}_n(t)(1 - t^2)^{k-1/2} dt / 2^k.$$

Proof. A formal calculation yields,

$$\int_{-1}^{1} \left\{ \sum_{n=0}^{\infty} \mathrm{H}_n(t) z^n \sum_{k=0}^{\infty} \alpha_k z^k 2^{-k} (1 - t^2)^{k-1/2} \right\} dt$$
$$= \sum_{n=0}^{\infty} z^n \left[\sum_{\nu=0}^{n} \alpha_\nu \int_{-1}^{1} \mathrm{H}_{n-\nu}(t) 2^{-\nu} (1 - t^2)^{\nu-1/2} dt \right].$$

By a comparison of coefficients, we have $\sum_{\nu=0}^{n} \alpha_\nu E_{n-\nu,\nu} = A_n$, which yields (9.16).

2. *Residue theorems*. There exists a simple method for the construction of functions of the class $\mathcal{C}(\mathrm{E})$ with certain singularities In fact suppose that $f(z)$ is a function, which is regular at the origin and possesses a singularity at the point α. For example, take $f(z) = (z - \alpha)^{-1}$. The function[15] $u(z, \bar{z}) = \mathrm{P}[(z - \alpha)^{-1}]$ will belong by definition (see §2) to the class $\mathcal{C}(\mathrm{E})$, and will be defined by the integral representation (2.6) over the domain $\mathfrak{B}^2 = E[|z| < \infty] - E[z = 2\alpha S,\ 1 \leq S < \infty]$. As we shall show immediately, $u(z, \bar{z})$ is also

(15) Sometimes it is useful, for the construction of functions with singularities to use operators slightly different from (2.6), for example, operators of the form $\mathrm{P}_1(f) = \mathrm{E}_1(z, \bar{z}) f(z) + \int \mathrm{E}_2(z, \bar{z}, t) f[p(z, t)] dt$, or involving double integrals. If $f(z)$ becomes infinite in such a way that the above integral is a regular function of $z, \bar{z}$, then $\mathrm{P}_1(f)$ possesses a singularity of the same character as f. (See also §1.) In particular, this method yields fundamental solutions.

Note that by the method indicated in [3, especially, p. 1173] various representations of this form can be easily obtained, E_1 and E_2 being solutions of certain integral or differential equations.

regular on $E[z=2\alpha S,\ 1<S<\infty]$. Let $z^0=Re^{i\varphi}$, $R=2|\alpha|S$, $S>1$, and $\varphi=\arg\alpha$. The function $[z^0((1-t^2)/2)-\alpha]^{-1}(1-t^2)^{-1/2}$ considered as a function of the complex variable $t=t_1+it_2$ possesses two poles, namely at the points $t=t^{(1)}\equiv+(1-S^{-1})^{1/2}$ and $t=t^{(2)}\equiv-(1-S^{-1})^{1/2}$. We write

$$\mathfrak{S}_1^1 = E[-1<t_1<1,\ t_2=0],$$

$$\mathfrak{S}_2^1 = E[-1<t_1<1,\ t_2=0] - \mathop{\mathrm{S}}_{k=1}^{2} E[-\epsilon+t^{(k)}<t_1<t^{(k)}+\epsilon,\ t_2=0]$$
$$+ \mathop{\mathrm{S}}_{k=1}^{2} E[t=t^{(k)}+\epsilon e^{i\varphi},\ 180^\circ \leqq \varphi \leqq 360^\circ],$$

$$\mathfrak{S}_3^1 = E[-1<t_1<1,\ t_2=0] - \mathop{\mathrm{S}}_{k=1}^{2} E[-\epsilon+t^{(k)}<t_1<t^{(k)}+\epsilon,\ t_2=0]$$
$$+ \mathop{\mathrm{S}}_{k=1}^{2} E[t=t^{(k)}+\epsilon e^{i\varphi},\ 0 \leqq \varphi \leqq 180^\circ],$$

ϵ being sufficiently small.

For every $z^{(1)}=Re^{i\psi}$, $\psi<\arg\alpha$, we have

$$\int_{\mathfrak{S}_1^1}[(z/2)(1-t^2)-\alpha]^{-1}(1-t^2)^{-1/2}dt = \int_{\mathfrak{S}_2^1}[(z/2)(1-t^2)-\alpha]^{-1}(1-t^2)^{-1/2}dt$$

since $\mathfrak{S}_1^1$ can be reduced to $\mathfrak{S}_2^1$ without cutting across singularities. The second integral exists, and represents an analytic function even for $z^{(1)}=z^{(0)}$, since the integrand is regular on $\mathfrak{S}_2^1$. It follows that $\int_{-1}^{1}[(z/2)(1-t^2)-\alpha]^{-1}(1-t^2)^{-1/2}dt$ is a regular function of z in the point $z^{(0)}$. However, in general we shall find different values for this function if we approach first from the left and then from the right, since $\mathfrak{S}_3^1$ cannot be reduced to $\mathfrak{S}_2^1$ without cutting the poles at $t^{(1)}, t^{(2)}$. Hence the point 2α will in general be a branch point of $P[(z-\alpha)^{-1}]$.

Suppose that $f(z)$ possesses a denumerable number of singularities which have no accumulation point within a finite distance of the origin. Then an analogous consideration shows that $P(f)$ will possess, at corresponding points, singularities which are, in general, branch points of $P(f)$. In that case, the integral formula (2.6) represents one branch of $P(f)$.

Now the problem of characterizing these singularities arises([16]). If the functions $u(z,\bar z)$ belonging to a class $\mathcal{C}(E)$ coincide with analytic functions of one complex variable, along certain curves, we may use this fact for one kind of characterization of singularities.

The procedure which can be applied may be demonstrated in the case where E satisfies the relation (9.5).

([16]) If $E(z,\bar z,t)$ satisfies certain differential equations, the function $P[(z-\alpha)^{-1}]$ satisfies *ordinary* differential equations, with coefficients which are connected with $E(z,\bar z,t)$ in a simple way. For details see [2].

Suppose at first that $u(z, \bar{z})$ is regular in the circle $\mathfrak{K}^2 = \mathrm{E}[|z| \leqq \rho]$. Then by Corollary 3.1, $f(z)$, (and hence also, by (9.3), $h(z)$) is regular in $\mathfrak{K}^2$, and we have

$$\int_{\mathfrak{k}^1} u(z, \bar{z}) dz = \int_{\mathfrak{k}^1} h(z) dz = 0, \qquad \mathfrak{k}^1 = \mathrm{E}[|z| = \rho]. \tag{9.17}$$

Thus under the conditions indicated above, the line integral (9.17) taken along a *circle* vanishes if $\mathfrak{k}^1$ lies in regularity domain of $u(z, \bar{z})$.

Suppose now that $f(z)$ has a pole, that is, say we have $f(z) = (z-\alpha)^{-1}$ and $\rho > 2\alpha$. Since the point 2α is a branch point, $\mathfrak{k}^1 = \mathrm{E}[|z| = \rho]$ will be now an *open* curve on the Riemann surface of $u(z, \bar{z})$. (It lies in the sheet in which the representation (2.5) is valid.) Both end points of $\mathfrak{k}^1$ lie on the slit

$$\mathrm{E}[z = 2\alpha S,\ 1 \leqq S < \infty]$$

(but of course in different sheets of the Riemann surface).

We have, then,

$$\begin{aligned} &\int_{\mathfrak{k}^1} \mathrm{P}[(z-\alpha)^{-1}] dz = 4\pi i \int_{t=t^{(2)}}^{t=t^{(1)}} \mathrm{E}_1(\rho, t)(1-t^2)^{-3/2} dt, \\ &\qquad t^{(n)} = -(-1)^n (1 - 2|\alpha|\rho^{-1})^{1/2}, \qquad n = 1, 2. \end{aligned} \tag{9.18}$$

Proof. The left-hand member of (9.18) can be written in the form

$$\int_{\mathfrak{k}^1} \int_{-1}^{1} \mathrm{E}_1(\rho, t)[(z/2)(1-t^2) - \alpha]^{-1}(1-t^2)^{-1/2} dt dz. \tag{9.19}$$

The integrand of (9.19) is an absolutely integrable function. We have therefore

$$\int_{\mathfrak{k}^1} \int_{t=-1}^{t=1} \cdots = \int_{\mathfrak{k}^1} \int_{t>t^{(2)}}^{t<t^{(1)}} \cdots + \int_{\mathfrak{k}^1} \int_{t>-1}^{t<t^{(2)}} \cdots + \int_{\mathfrak{k}^1} \int_{t>t^{(1)}}^{t<1} \cdots.$$

Changing the order of integration, the residue theorem then gives

$$\int_{\mathfrak{k}^1} \frac{\mathrm{E}_1(\rho, t) dz}{[(z/2)(1-t^2) - \alpha](1-t^2)^{1/2}} = \begin{cases} 4\pi i \dfrac{\mathrm{E}_1(\rho, t)}{(1-t^2)^{3/2}} & \text{for} \quad t^{(2)} < t < t^{(1)}, \\ 0 & \text{for} \quad -1 < t < t^{(2)}, \\ 0 & \text{for} \quad t^{(1)} < t < 1, \end{cases}$$

which yields (9.18).

The analogous formula for $\mathrm{P}[(z-\alpha)^{-n}]$, $n > 1$, an integer, can be obtained in the following way: The integral, in a certain neighborhood of every point $\alpha = \alpha^0$ for which $|\alpha^0| \neq \rho$ is a regular function of α.

Differentiating (9.18) n times with respect to α we obtain

$$(9.20) \qquad (-1)^n n! \int_{\mathfrak{f}^1} \mathrm{P}[(z-\alpha)^{-(n+1)}]dz = 4\pi i \frac{d^n}{d\alpha^n}\left\{ \int_{t=t^{(2)}}^{t=t^{(1)}} \frac{\mathrm{E}_1(\rho, t)dt}{(1-t^2)^{3/2}} \right\}.$$

(Note that α appears only in $t^{(1)}$ and $t^{(2)}$.)

10. **A connection with a class of difference equations.** There exists an important connection between differential and difference equations. In particular, some of our previous results can be used for the theory of difference equations of the type[17]

$$(10.1) \qquad \begin{aligned} &(M+1)(N+1)\psi(M+1, N+1) \\ &\quad + \sum_{S=0}^{S_1}\sum_{K=0}^{K_1} \alpha_{SK}(M-S+1)\psi(M-S+1, N-K) \\ &\quad + \sum_{S=0}^{S_2}\sum_{K=0}^{K_2} \beta_{SK}(N-K+1)\psi(M-S, N-K+1) \\ &\quad + \sum_{S=0}^{S_3}\sum_{K=0}^{K_3} \gamma_{SK}\psi(M-S, N-K) = 0. \end{aligned}$$

Here α_{SK}, β_{SK}, and γ_{SK} are constants.

THEOREM 10.1. *Let*

$$(10.2) \qquad U(z, \bar{z}) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \psi(m, n) z^m \bar{z}^n$$

be a solution of (1.1), *where*

$$(10.3) \qquad a = \sum_{s=0}^{S_1}\sum_{k=0}^{K_1} \alpha_{sk} z^s \bar{z}^k, \qquad b = \sum_{s=0}^{S_2}\sum_{k=0}^{K_2} \beta_{sk} z^s \bar{z}^k, \quad \textit{and} \quad c = \sum_{s=0}^{S_3}\sum_{k=0}^{K_3} \gamma_{sk} z^s \bar{z}^k.$$

Then ψ (M, N) is a solution of the difference equation (10.1).

Proof. We have

$$U_{z\bar{z}} = \sum_M \sum_N (M+1)(N+1)\psi(M+1, N+1) z^M \bar{z}^N,$$

$$U_z \sum_{S=0}^{S_1}\sum_{K=0}^{K_1} \alpha_{SK} z^S \bar{z}^K = \sum_M\sum_N\sum_{S=0}^{S_1}\sum_{K=0}^{K_1} \alpha_{SK}(M-S+1)\psi(M-S+1, N-K) z^M \bar{z}^N,$$

$$U_{\bar{z}} \sum_{S=0}^{S_2}\sum_{K=0}^{K_2} \beta_{SK} z^S \bar{z}^K = \sum_M\sum_N\sum_{S=0}^{S_2}\sum_{K=0}^{K_2} \beta_{SK}(N-K+1)\psi(M-S, N-K+1) z^M \bar{z}^N,$$

$$U \sum_{S=0}^{S_3}\sum_{K=0}^{K_3} \gamma_{SK} z^S \bar{z}^K = \sum_M\sum_N\sum_{S=0}^{S_3}\sum_{K=0}^{K_3} \gamma_{SK}\psi(M-S, N-K) z^M \bar{z}^N.$$

[17] We note that the difference equations in two variables have been treated very little by analytical methods. As far as I know the only results in this direction were obtained by C. R. Adams, *On the existence of solutions of a linear partial pure difference equation*, Bull. Amer. Math. Soc. vol. 32 (1926) p. 197.

Since U is supposed to satisfy equation (1.1), the function $\psi(M, N)$ must satisfy (10.1).

REMARK. In speaking of a solution, $\psi(M, N)$ of (10.1), we shall in the future always suppose that the $\psi(M, N)$ have the following property: There exists a number $\rho > 0$ such that $\sum_{M=0}^{\infty}\sum_{N=0}^{\infty}|\psi(M, N)|\rho^{M+N} < \infty$. Under this hypothesis, it follows that conversely, to every solution $\psi(M, N)$ of (10.1) there corresponds a function $U(z, \bar{z})$ given by (10.2) and satisfying $L(U) = 0$.

The connection indicated in Theorem (10.1) enables one to reduce many problems of the theory of difference equations of type (10.1) to that of functions satisfying $L(U) = 0$. Then the application of the methods of the theory of partial differential equations may yield the desired result.

As an example of such a procedure, the following problem may be considered:

To give a representation of the solution $\psi(M, N)$ of (10.1) in terms of $\psi(N, 0)$ and $\psi(0, N)$, $n = 0, 1, 2, \cdots$.

By Theorem (10.1) (cf. also the remark following it) this problem is equivalent to finding the coefficients of U satisfying (1.1), where the functions a, b and c are given by (10.3). On the other hand, by (7.6) we have for the coefficients A_{mn} the relation

$$\psi(M, N) =$$

$$- \frac{1}{4\pi^2 M! N!} \int_0^{2\pi} \int_0^{2\pi} \int_{-1}^{1} \sum_{k=1}^{2} \mathrm{H}_k \mathrm{E}_k f_k d\phi_1 d\phi_2 dt \Big/ r_1^M r_2^N e^{i(M\phi_1 + N\phi_2)}(1 - t^2)^{1/2}, \tag{10.4}$$

Here

$$H_1 = \exp\left[- \int_0^{(r_1 e^{i\phi_1} - i r_2 e^{i\phi_2})} a(r_1 e^{i\phi_1} + i r_2 e^{i\phi_2}, \bar{z}) d\bar{z} \right],$$

$$H_2 = \exp\left[- \int_0^{(r_1 e^{i\phi_1} + i r_2 e^{i\phi_2})} b(z, r_1 e^{i\phi_1} - i r_2 e^{i\phi_2}) dz \right],$$

$$f_1 = \sum_{M=0}^{\infty} \frac{\psi(M, 0)}{M!} \left[\frac{(r_1 e^{i\phi_1} + i r_2 e^{i\phi_2})(1 - t^2)}{2} \right]^M,$$

$$f_2 = \sum_{N=0}^{\infty} \frac{\psi(0, N)}{N!} \left[\frac{(r_1 e^{i\phi_1} - i r_2 e^{i\phi_2})(1 - t^2)}{2} \right]^N,$$

and the functions $\mathrm{E}_k = \mathrm{E}_k(r_1 e^{i\phi_1} + i r_2 e^{i\phi_2},\ r_1 e^{i\phi_1} - i r_2 e^{i\phi_2},\ t)$, $k = 1, 2$, are generating functions of the totality of functions satisfying (1.1) when a, b, c are given by (10.3).

The representation (10.4) enables us to draw various conclusions concerning $\psi(M, N)$. For instance, the growth properties of $\psi(M, N)$ (considered as a function of M and N), in terms of the growth properties of $\psi(N, 0)$ and $\psi(0, N)$, can be obtained from this relation.

BIBLIOGRAPHY

1. Stefan Bergman, *Anwendung des Schwarzschen Lemmas auf eine Klasse von Funktionen von zwei komplexen Veränderlichen*, Jber. Deutschen Math. Verein. vol. 39 (1930) pp. 163–168.

2. ———, *Sur un lien entre la théorie des équations aux derivées partielles elliptiques et celle des fonctions d'une variable complexe*, C. R. Acad. Sci. Paris vol. 205 (1937) pp. 1198–1200 and 1360–1362.

3. ———, *Zur Theorie der Funktionen, die eine lineare partielle Differentialgleichung befriedigen* Rec. Math. (Mat. Sbornik) N.S. vol. 2 (1937) pp. 1169–1198.

4. ———, *The approximation of functions satisfying a linear partial differential equation*, Duke Math. J. vol. 6 (1940) pp. 537–561.

5. ———, *Boundary values of functions satisfying a linear partial differential equation of elliptic type*, Proc. Nat. Acad. Sci. U.S.A. vol. 26 (1940) pp. 668–671.

6. ———, *Ueber eine Integraldarstellung von Funktionen zweier komplexer Veränderlichen*, Rec. Math. (Mat. Sbornik) N.S. vol. 1 (1936) pp. 851–862.

7. ———, *Sur les fonctions orthogonales de plusieurs variables complexes*, Interscience Publishers, New York, 1941.

8. G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals*. II, Math. Zeit. vol. 34 (1932) pp. 403–439.

9. D. Hilbert, *Grundzüge einer allgemeiner Theorie der linearer Integralgleichungen*, Leipzig and Berlin, 1912.

10. N. Nielsen, *Handbuch der Theorie der Gammafunktionen*, Leipzig, 1906.

11. A. Sommerfeld *Randwertaufgaben in der Theorie der partiellen Differentialgleichungen*. Enzyklopädie der mathematischen Wissenschaften, vol. II, A, 7c pp. 504–570.

12. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications vol. 20, 1935.

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